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N Body Spatial Parabolic Orbits Asymptotic to Collinear Central Configurations

CLARK ROBINSON AND DONALD G. SAARI*

Department of Mathematics, Northwestern University, Evanston, Illinois 60201

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1. INTRODUCTION

An orbit for the Newtonian n body problem is called *Parabolic* when all the mutual distances between particles approach infinity while the velocities tend to zero as time t goes to infinity. The asymptotic growth rate for these mutual distances turns out to be constant multiples of t raised to the $2/3$ power (or to the power $2/(p+1)$ for the inverse p central force law, $1 < p < 3$) [11]. Indeed, when the position vectors are scaled by dividing by $t^{2/3}$, or by the square root of the moment of inertia, then the scaled position vectors tend to the set of central configurations. (A central configuration is formed whenever for all the particles a fixed constant multiple of the mass times the position vector is equal to the corresponding force vector.)

It turns out that central configurations can be identified with the critical point set of a certain function. Then, the configurations are classified as being either degenerate or nondegenerate to correspond with the classification of the critical point of this function. Should the central configuration be nondegenerate, as is known to be the case for collinear central configurations, the equilateral triangle configuration for the three body problem, and the equilateral tetrahedron configuration for the four body problem, then the limiting configuration of a given motion must be unique. (For a discussion of central configurations, see [15, Chap. 5; 8; 12]). This uniqueness follows by combining two facts. First, the nondegeneracy implies that the orbit of the central configuration by the $SO(3)$ action must be isolated in the set of all central configurations. (The $SO(3)$ action on configuration space is rotation about the center of mass of the system.) Secondly, there can be no rotation of the limiting configuration in the sense of motion along the $SO(3)$ action on configuration space. (See [7] for the planar three body problem and [13] for the general case. The lack of

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rotation of the limiting configuration is proved in the general case even should the configuration be degenerate, but it is not known whether the uniqueness of the limiting configuration is true in general.)

The purpose of this paper is to study in greater detail the behavior of particles as they approach a limiting collinear central configuration. In particular, we will be interested in determining how the mass ratios of the particles effect the rates at which the scaled position vectors approach the limiting configuration. We will show that *these scaled vectors asymptotically approach the limiting position like* $t^{-1/6}\log t$ *or like* t^b *for some* b *depending upon the mass ratio and satisfying* $-1/3 < b < 0$. This means that the *mutual distances between particles* (given by the unscaled equations) *must separate like* $At^{2/3} + B(1+o(1))\,t^{1/2}\log t$ *or like* $At^{2/3} + B(1+o(1))\,t^{b+2/3}$, where A and B denote positive contants for which their values may change with the choice of the indices of the particles. For example, constants A are determined in the following manner. The scaled equations approach a central configuration, so each particle approaches some one vertex. The distances between appropriate vertices yield the appropriate values for A.

Next we shall investigate the role angular momentum coupled with the mass ratio plays in these expansion rates. The results are similar, but the restriction of b now becomes $-1/6 \leqslant b < 0$. Although we shall concentrate on the collinear central configuration situation, obvious modifications of the approach employed here yield similar results for other limiting nondegenerate central configurations.

In the study of three colliding particles, Painleve [15] raised the question whether it was possible for the colliding particles to approach the collision point by entering into an infinite spin. The same question arises for parabolic orbits [11]; namely, as t approaches infinity can the particles rotate an infinite number of times as they approach the limiting configuration. The interpretation of this question as presented in [15] and as answered in [13] was whether or not the scaled position vectors whould have a limiting position. It turns out that for "total collapse" collisions, this is sufficient to resolve the issue. This is because, as a corollary to the analysis in [13], it was shown that the properties of the limiting configuration impose strict restraints on the manifold of orbits tending toward these configurations. In particular, should the limiting configuration be collinear, then for all time the solution exists, the motion was collinear. But collinear motion cannot admit the spinning effects described by the Painleve–Wintner problem. A similar statement holds for higher dimensional limiting configurations.

The constraints on the manifold of parabolic motion are not as severe as for collisions; for example, there exist collinear parabolic orbits which tend to collinear central configurations. It is this flexibility admitted by the extra dimension which may permit a spin in "lower order" terms; if this is the case, then the approach used in [13] would provide only a first order answer.

To see why, consider the vector $t^{2/3}\mathbf{I} + t^{1/2}\cos t\mathbf{J} + t^{1/2}\sin t\mathbf{K}$. The scaled equation approaches the fixed vector position $\mathbf{I}$, but the unscaled vector has an oscillation which is expanding infinitely far from the origin.

In this paper we extend the Painleve–Wintner question by asking whether the differential equations admit an infinite physical rotation of the system in the sense that the system rotates an infinite number of times about the axis of the limiting configuration. Since the limiting configuration cannot rotate, it is obvious that this type of result can only occur should the limiting configuration be collinear. We answer this extension of the Painleve-Wintner question by showing that *for some mass ratios the system does make an infinite number of rotations in a logarithmic fashion while for other mass ratios the angle of rotation is finite.* It was the possible existence of this infinite spin about the z-axis which kept Hulkower [7] from extending his results to the full three-dimensional three body problem.

In Section 2 the basic ideas and results are illustrated in the special case of the isosceles three body problem. In this simplified setting, the asymptotic rates can be calculated explicitly in terms of the mass ratios. In Section 3, the general n body problem is treated using the scaling of the configuration by $t^{2/3}$ as developed in [13]. Finally, in Section 4 we indicate how the results could have been derived by use of another scaling given by the McGehee coordinates as developed in [2]. In these coordinates the configuration is scaled by dividing by the square root of the moment of inertia.

2. Isosceles Three Body Problem

The isosceles three body problem is a special case of the general three body problem where, because of symmetry in the choice of the masses and the initial conditions, the solution forms an isosceles triangle for all time the solution exists. The following elementary statement describes the symmetry assumptions which are necessary for this to hold.

Proposition. *Let $T: R^n \to R^n$ be a linear mapping which has an invariant subspace $A \subset R^n$. For the system of differential equations*

$$\mathbf{r}_i'' = \mathbf{F}_i(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \qquad i = 1, 2, 3, \tag{2.1}$$

assume that the $\mathbf{F}$*'s are such that whenever $T(\mathbf{r}_1) = \mathbf{r}_3$ and $\mathbf{r}_2 \in A$, then $T\mathbf{F}_1 = \mathbf{F}_3$ and $\mathbf{F}_2 \in A$. Furthermore, assume that the* $\mathbf{F}$*'s are sufficiently smooth to ensure that this system has a unique solution. Then for initial conditions $T(\mathbf{r}_1(0)) = \mathbf{r}_3(0)$, $T(\mathbf{r}_1'(0)) = \mathbf{r}_3'(0)$, and both $r_2(0)$, $\mathbf{r}_2'(0) \in A$, the solution satisfies $T(\mathbf{r}_1((t) = \mathbf{r}_3(t)$ and $\mathbf{r}_2(t) \in A$ for all values of t such that the solution exists.*

Proof. Consider the system of related equations

$$\mathbf{r}_i'' = \mathbf{F}_i(\mathbf{r}_1, \mathbf{r}_2', T(\mathbf{r}_1)), \qquad i = 1, 2,$$

with initial conditions $\mathbf{r}_2(0)$, $\mathbf{r}_2'(0) \in A$. According to the smoothness assumption on the F's, a unique solution exists to this system. The assumptions on the initial conditions, $\mathbf{F}_2$, and the fact that $\mathbf{r}_3$ is replaced by $T(\mathbf{r}_1)$ provide that $\mathbf{r}_2(t) \in A$. By direct substitution of this solution into Eq. (2.1), where we let $\mathbf{r}_3 = T(\mathbf{r}_1)$, we have that both the equations and the initial conditions are satisfied. The conclusion of the proposition follows from the uniqueness of the solutions.

We now return to the isosceles triangle solutions. First we consider the planar problem with no angular momentum. Let $T: R^2 \to R^2$ be the involution mapping which reflects vectors with respect to the y-axis. For this mapping, the invariant subspace A is the y-axis. According to the proposition, if the forces and the initial conditions satisfy the appropriate conditions, then the second particle will always remain on the y axis and the particles will form an isosceles triangle. To achieve the condition on the forces, it suffices to assume that the two masses $m(1)$ and $m(3)$ are equal: say $m(1) = m(3) = m$. The remaining mass $m(2)$ is arbitrary, but as a way of normalizing the system we assume that $m(2) = 1$. Assume that the origin is fixed at the center of mass of the system. For initial conditions, assume that particles 1 and 3 are symmetrically located with respect to the y axis at (x, y) and $(-x, y)$. Thus, particle 2 is initially at $(0, -2my)$. (See Figure 1.) By restricting the initial velocities to satisfy the appropriate symmetry conditions (see the proposition), the solution forms a coplanar isosceles triangle for all time.

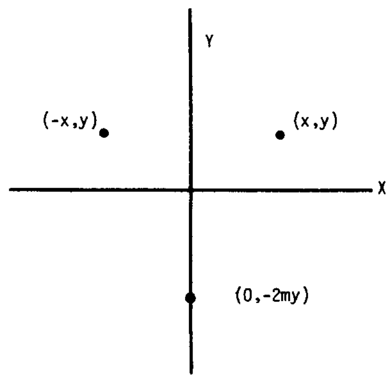

FIGURE 1

Denoting $\mu = 1 + 2m$, the equations of motion are

$$\begin{aligned} \ddot{x} &= -x/(x^2 + \mu^2 y^2)^{3/2} - m/4x^2, \\ \ddot{y} &= -\mu y/(x^2 + \mu^2 y^2)^{3/2}. \end{aligned} \tag{2.1}$$

We use the fact that parabolic motion expands like $t^{2/3}$ to scale the system by introducing the time dependent change of variables $x = t^{2/3}\xi$ and $y = t^{2/3}\eta$. The equations of motion becomes

$$\begin{aligned} \ddot{\xi}t^{2/3} &= -\tfrac{4}{3}\dot{\xi}t^{-1/3} + \tfrac{2}{9}\xi t^{-4/3} - \xi t^{-4/3}/(\xi^2 + \mu^2\eta^2)^{3/2} - (m/4)/(\xi^2 t^{4/3}), \\ \ddot{\eta}t^{2/3} &= -\tfrac{4}{3}\dot{\eta}t^{-1/3} + \tfrac{2}{9}\eta t^{-4/3} - \mu\eta t^{-4/3}/(\xi^2 + \mu^2\eta^2)^{3/2}. \end{aligned} \tag{2.2}$$

Theorem 1 characterizes the behavior of this system in terms of the value of m. It turns out that this system cannot admit a Painleve–Wintner type spin. On the other hand, for certain mass values it does admit an oscillation in that the particles pass through the x-axis an infinite number of times.

THEOREM 1. *Consider the planar isosceles three body problem with a fixed axis as given by Eqs.* (2.1). *Assume that the particles are in parabolic motion and that the motion approaches a collinear central configuration as time t approaches infinity. Then the distances of the particles from the axis either is zero or grows as given in the following table*

m	*Scaled equations*	*Unscaled equations*
$> \frac{4}{55}$	$t^{-1/6}$	$t^{1/2}$
$= \frac{4}{55}$	$t^{-1/6}$ *or* $t^{-1/6}\log t$	$t^{1/2}$ *or* $t^{1/2}\log t$
$< \frac{4}{55}$	t^{a_i} *where*	t^{b_i} *where*
	$-\frac{1}{3} < a_1 < -\frac{1}{6} < a_2 < 0$	$\frac{1}{3} < b_1 < \frac{1}{2} < b_2 < \frac{2}{3}$

The motion is oscillatory, in the sense that each particle passes through the x-axis an infinite number of times, if and only if $m > \frac{4}{55}$.

A more precise asymptotic representation of the motion is given in the proof. In particular, see Eqs. (2.4)–(2.8).

To prove this theorem we recognize that after multiplying both sides of Eq. (2.2) by $t^{4/3}$, the resulting equations form an Euler system of differential equations. Thus the standard change of independent variables $\ln(t) = s$ converts the system to the following autonomous system, where ($'$) denotes differentiation with respect to s:

$$\begin{aligned} \xi'' &= -\tfrac{1}{3}\xi' + \tfrac{2}{9}\xi - \xi/(\xi^2 + \mu^2\eta^2)^{3/2} - m/4\xi^2, \\ \eta'' &= -\tfrac{1}{3}\eta' + \tfrac{2}{9}\eta - \mu\eta/(\xi^2 + \mu^2\eta^2)^{3/2}. \end{aligned} \tag{2.3}$$

The two equilibrium solutions for these equations are

$$\xi(L) = (9(4+m)/8)^{1/3}, \quad \eta(L) = 0 \quad \text{(collinear configuration)}$$

or

$$\xi(E) = (9\mu/16)^{1/3}, \quad \eta(E) = 3^{1/2}(9/16\mu^2)^{1/3} \quad \text{(equilateral triangle solution)},$$

where $\xi' = \eta' = 0$.

These equilibrium solutions correspond to solutions of Eq. (2.1) which are fixed constant multiples of $t^{2/3}$. It turns out that a solution is parabolic if and only if the orbit is asymptotic to one of these equilibrium solutions [13]. Therefore, the asymptotic behavior of a parabolic orbit can be determined by analyzing these equilibrium points. This is the method we use to prove Theorem 1. Indeed, it will turn out that the a's and the b's which appear as exponents in the statement of the theorem correspond to the eigenvalues resulting from the associated study of the linearized equations.

Proof of Theorem. When Eqs. (2.3) are linearized about the collinear configuration they determine a 4×4 matrix. Because $\eta(L) = 0$, all entries not in the upper left-hand and the lower right-hand 2×2 submatrices are zero. Thus this analysis reduces to the submatrices

$$\begin{pmatrix} 0 & 1 \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

in the (ξ, ξ') variables and

$$\begin{pmatrix} 0 & 1 \\ \frac{2}{9} - \mu\xi^{-3}(L) & -\frac{1}{3} \end{pmatrix}$$

in the (η, η') variables. The eigenvalues for the first matrix are -1 and $\frac{2}{3}$ with eigenvectors $(1, -1, 0, 0)$ and $(1, -\frac{2}{3}, 0, 0)$, respectively. This corresponds to motion along the axis $\eta = 0$.

The eigenvalues of the second submatrix satisfy the equation

$$\lambda^2 + \tfrac{1}{3}\lambda - \tfrac{2}{9} + \mu\xi^{-3}(L) = 0$$

or

$$\lambda = -\tfrac{1}{6} \pm (\tfrac{1}{6})\{9 - (32\mu/(4+m))\}^{1/2}$$

Let $f(m) = 9 - (32\mu/(4+m)) = (4 - 55m)/(4+m)$ be the discriminant. Note that $f(0) = 1$, $f(m) \to -55$ as m approaches infinity, and that f is a monotonic function of m which has its unique zero at $m = \frac{4}{55}$. Thus, for

$m < \frac{4}{55}$, the eigenvalue are distinct, real, and negative and they satisfy the inequality $-\frac{1}{3} < \lambda(1) < -\frac{1}{6} < \lambda(2) < 0$. The corresponding eigenvectors are $(0, 0, 1, \lambda(i))$. If $m > \frac{4}{55}$, then the eigenvalues are complex conjugates with real part equal to $-\frac{1}{6}$. If $m = \frac{4}{55}$, then the eigenvalue $-\frac{1}{6}$ has multiplicity 2, and the matrix is not diagonalizable. The eigenvector is $(0, 0, 6, -1)$.

We now turn to the behavior of the solutions of the nonlinear equations. According to the stable manifold theorem for an equilibrium point and the above analysis of the linearized equations, there is a three-dimensional stable manifold of solutions tending to each collinear configuration. The collinear solutions form a one-dimensional submanifold with the eigenvalue -1. Since collinear solutions must remain collinear, we have that the solutions to the scaled equations are

$$\xi(s) = \xi(L) + B(1 + o(1)) \exp(-s), \qquad \eta(s) = 0. \tag{2.4a}$$

Here B is some scalar, $\xi(L)$ is the equilibrium solution discussed above, and $o(1)$ designates terms which tend to zero as s goes to infinity. The solution to the unscaled equations are

$$x(t) = \xi(L)\, t^{2/3} + B(1 + o(1))\, t^{-1/3}, \qquad y(t) = 0. \tag{2.4b}$$

The above shows that the collinear solutions form only a one-dimensional submanifold of the total three-dimensional manifold. Therefore, most of the solutions are *not* collinear, and their convergence rates are governed by the eigenvalues $\lambda(1)$ and $\lambda(2)$. (This is because these rates are slower than the convergence due to the eigenvalue -1.) This slower rate of convergence has $\eta(s)$, $\eta'(s)$ converging like $\exp(as)$ where a is the real part of one of the two eigenvalues. Thus if $m < \frac{4}{55}$, there is a two-dimensional manifold where, by application of the Hartman C^1 linearization theorem [5], the behavior of the scaled equations are

$$\eta(s) = C(1 + o(1)) \exp(as). \tag{2.5a}$$

(The Hartman result asserts that since the original equations are at least twice continuously differentiable, when the equations are restricted to the stable manifold they are C^1 conjugate to the linearized equations.) Here, C is an arbitrary scalar and $a = \lambda(1)$ is the smaller of the two eigenvalues, so a lies between the values $-\frac{1}{3}$ and $-\frac{1}{6}$.

When the equations of motion for ξ are expanded in a Taylor Series expansion about the central configuration, we obtain

$$\xi'' = -\tfrac{1}{3}\xi' + \tfrac{2}{3}(\xi - \xi(L)) + E\eta^2 + \text{h.o.t.}$$

where constant $E = 3\mu^2/\xi^4(L)$ and h.o.t. denotes higher order terms in ξ and

η. Substituting the result of (2.5a) into this equation converts the last term into $CE\exp(2as)$. Thus, this equation is a subsystem of the system

$$\xi'' = -\tfrac{1}{3}\xi' + \tfrac{2}{3}(\xi - (\xi(L)) + Ed + \text{h.o.t.},$$
$$d' = 2ad,$$

where the appropriate initial conditions are imposed upon d and where the h.o.t. differ from those of the first expression for ξ''. Invoking the stable manifold theorem about the critical point $\xi = \xi(L)$, $\xi' = 0$, $d = 0$, yields that

$$\xi = B(1 + o(1))\exp(2as), \tag{2.5b}$$

where B is a constant.

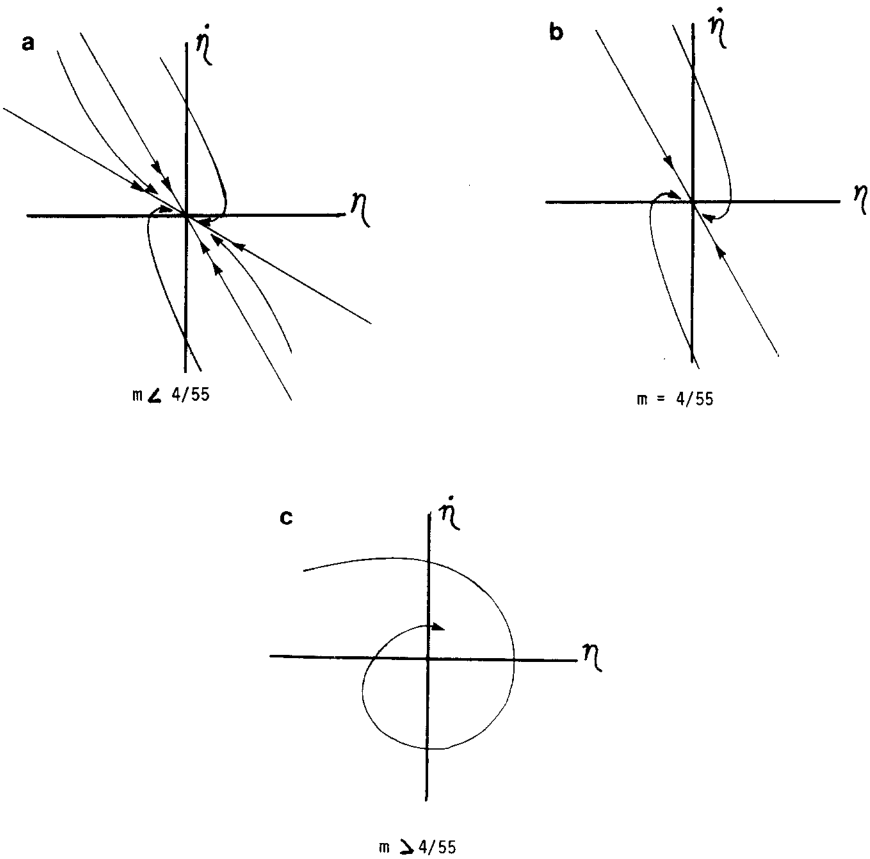

FIGURE 2

The solutions for the unscaled equations are

$$x(t) = \xi(L)\, t^{2/3} + B(1 + o(1))\, t^{2/3+a}, \qquad y(t) = C(1 + o(1))\, t^{a}, \quad (2.5c)$$

and the derivatives of $x(t)$ and $y(t)$ are precisely what you would obtain by formally differentiating these equations. (Differentiation is not permitted because the $o(1)$ term corresponds to an inequality, but the conclusion follows from our analysis of the linearized equations and the resulting eigenvectors.) Off of this strong-stable manifold, the solutions are as given by Eqs. (2.5), except a now equals $\lambda(2)$, so it lies between $-\frac{1}{6}$ and 0. For the relationship between these solutions, see Fig. 2.

If $m > \frac{4}{55}$, then from the stable manifold theorem and by using an argument similar to that used to obtain (2.5b), we find that

$$\eta(s) = C(1 + o(1)) \exp(-s/6) \cos(bs + D) \quad \text{and} \quad \xi(s), \quad \xi'(s)$$
$$\text{approach } \xi(L) \text{ and zero, respectively, like } \exp(-s/3), \qquad (2.6a)$$

where the capital letters correspond to scalars and b is the imaginary part of the eigenvalues. The unscaled solutions are

$$v(t) = C(1 + o(1))\, t^{1/2} \cos(b \ln t + D)$$

and (2.6b)

$$x(t) = \xi(L)\, t^{2/3} + O(t^{1/3}).$$

Notice that the solutions in Eqs. (2.6) cross the x-axis an infinite number of times in a logarithmatic fashion.

Finally we return to the case $m = \frac{4}{55}$. In this case eigenvalue $-\frac{1}{6}$ has multiplicity 2 and the matrix is not diagonalizable. Again we describe the motion by appealing to a combination of the stable manifold theorem and the Hartman result. From this we find that both $\eta(s)$ and $\eta'(s)$ approach zero like $\exp(-s/6)$ or like $s \exp(-s/6)$. Along a two-dimensional submanifold, we have that

$$\begin{aligned} \eta(s) &= B(1 + o(1)) \exp(-s/6), \\ \xi(s) &= \xi(L) + C(1 + o(1)) \exp(-s/3); \end{aligned} \qquad (2.7a)$$

and

$$\begin{aligned} y(t) &= B(1 + o(1))\, t^{1/2}, \\ x(t) &= \xi(L)\, t^{2/3} + C(1 + o(1))\, t^{1/3}. \end{aligned} \qquad (2.7b)$$

Off of this manifold, the behavior is

$$\eta(s) = B(1 + o(1))s \exp(-s/6),$$
$$\xi(s) = \xi(L) + C(1 + o(1))\, s^2 \exp(-s/3); \tag{2.8a}$$

or

$$y(t) = B(1 + o(1))\, t^{1/2} \ln(t),$$
$$x(t) = \xi(L)\, t^{2/3} + C(1 + o(1))\, t^{1/3}(\ln t)^2. \tag{2.8b}$$

See Fig. 2 for a representation of these motions.

Next we will look at the spatial three body problem. There are three natural involution mappings to investigate, and each gives rise to a special class of three body problems; in all three cases the map is represented by a 3×3 diagonal matrix. In the first case, all three nonzero entries are (-1). This map corresponds to a reflection about the origin, and the orbit one would obtain by appealling to the proposition would be a collinear orbit (hence, it is restricted to some plane for all time), where the second particle is fixed at the origin. These orbits are not difficult to analyze, and we do not do so here because they offer nothing new toward the theme of this paper.

The next two maps can be viewed as being different extensions of the involution used in the coplanar problem. First there is the map which has two of the entries equal to (-1) and the third, the middle one, is equal to unity. In this case the mapping is a reflection with respect to the y-axis, and the y-axis is the invariant subspace with respect to the mapping. The condition on the masses so that the forces will satisfy the conditions stated in the proposition is that $m(1) = m(3)$. The special solutions corresponding to the proposition are isosceles triangles. However, these solutions turn out to be special cases of what will be described in the next section, so we will not discuss them here.

Finally, we come to a map which, in conjunction with the proposition, does introduce a different behavior. This is the diagonal map which has two of the entries equal to 1 and the third equal to (-1). In order to facilitate comparison and to make this an extension of the planar problem, we will assume the (-1) entry is the first one. Thus this mapping corresponds to a reflection with respect to the y–z plane. Again, the condition imposed upon the masses in order to satisfy the "force" condition of the proposition is that $m(1) = m(3) = m$ while $m(2) = 1$. The solutions assured by the proposition form isosceles triangles with particle 2 in the invariant subspace (of the involution map), which is the y–z plane.

The extra degree of freedom is represented by the z-axis, and to obtain the scaled equations we define $\rho = zt^{-2/3}$. The equations of motion for the scaled variables after the Euler change of independent variables has been made are

$$\begin{aligned}
\xi'' &= -\tfrac{1}{3}\xi' + \tfrac{2}{9}\xi - \xi/(\xi^2 + \mu^2\eta^2 + \mu^2\rho^2)^{3/2} - m/4\xi^2, \\
\eta'' &= -\tfrac{1}{3}\eta' + \tfrac{2}{9}\eta - \mu\eta/(\xi^2 + \mu^2\eta^2 + \mu^2\rho^2)^{3/2}, \\
\rho'' &= -\tfrac{1}{3}\rho' + \tfrac{2}{9}\rho - \mu\rho/(\xi^2 + \mu^2\eta^2 + \mu^3\rho^2)^{3/2}.
\end{aligned} \tag{2.9}$$

For this system, any collinear solution must lie on the x-axis. Thus the unique collinear central configuration corresponds to $\xi(L) = \{8/9(4+m)\}^{1/3}$ and $\eta(L) = \rho(L) = 0$.

The new feature we wish to study in this setting is the effect of the mass ratio upon possible rotation about the axis of the limiting configuration and the concomitant restriction imposed upon the asymptotic growth rate. Before we present a statement of the results, recall that if a system of three bodies has zero angular momentum, then the motion is confined to lie in a fixed plane for all time [9]. The symmetry assumptions force this plane to contain the x-axis, so for zero angular momentum the analysis reduces to the above discussed coplanar problem. Furthermore, it follows from the symmetry conditions imposed on the initial conditions (to apply the proposition) that if these special solutions are coplanar, then the angular momentum must be zero. As a result of this, the extension of the solutions from the coplanar to the spatial setting can be viewed as being a manifestation of introducing nonzero angular momentum.

THEOREM 2. *Consider the solutions to the spatial isosceles three body problem with fixed axis as given by scaled equations* (2.9). *The behavior of any coplanar solution is given in Theorem* 1. *For any solution which is noncoplanar, the asymptotic growth rate is given in the table below. In particular, the solution oscillates infinitely often about the x-axis if and only if $m > \frac{4}{55}$. If there is such oscillation, then the solution does so in a logarithmic fashion.*

Mass	*Scaled equations*	*Unscaled behavior*
$m < \frac{4}{55}$	t^a, *where* $-\frac{1}{6} < a < 0$	t^b, *where* $\frac{1}{2} < b < \frac{2}{3}$
$m = \frac{4}{55}$	$t^{-1/6}\log t$	$t^{1/2}\log t$
$m > \frac{4}{55}$	$t^{-1/6}$	$t^{1/2}$

Proof. Since the coplanar situation reduces to that of Theorem 1, we shall be interested in the effects coming from the noncoplanar solutions. The eigenvalues resulting from the linearized part of Eq. (2.9) evaluated about the collinear central configuration are the same as in the planar case with the only difference being that $\lambda(1)$ and $\lambda(2)$ each have multiplicity two. As a result, the possible asymptotic growth rates of the distance to the axis of the limiting configuration is that same as in the planar case. Furthermore, with these five eigenvalues it follows from the stable manifold theorem for an equilibrium point that these solutions for the spatial problem form a five-dimensional manifold. As we have shown above, for a fixed plane the coplanar solutions form a three-dimensional submanifold. The fixed plane must contain the x-axis, which means that the plane is determined by its intersection with the y–z plane. Therefore, the set of all coplanar solutions forms a four-dimensional submanifold of this five-dimensional manifold.

For $m < \frac{4}{55}$, the eigenvalue -1 corresponds to the c direction and the eigenvector is $(1, -1, 0, 0, 0, 0)$. Corresponding to eigenvalue $\lambda(i)$ with multiplicity 2 are the eigenvectors $(0, 0, 1, \lambda(i), 0, 0)$ and $(0, 0, 0, 0, 1, \lambda(i))$. As explained above, any coplanar solution is determined by the orientation of the plane of motion, which in turn is determined by how it intersects in the y–z coordinate plane. The eigenvalues for this planar motion are as determined in the proof of Theorem 1, and the only change in the eigenvectors is that they must be adjusted to correspond to the new orientation of the plane. In other words, the eigenvectors will correspond to a fixed combination of the pair of eigenvectors given above. This then characterizes, at least for the linear equations, the behavior of the four-dimensional manifold corresponding to coplanar solutions. It is now a simple linear algebra exercise to establish that any noncoplanar solution must have nonzero projection on eigenspace determined by $\lambda(2)$, where $-\frac{1}{6} < \lambda(2) < 0$. Since this corresponds to the slower growth rate, it will dominate in the asymptotic description.

Again, by use of the stable manifold theorem for an equilibrium point and a Hartman theorem, the above analysis establishes the asserted growth rates of the solution. An argument similar to that used in the proof of Theorem 1 obtains asymptotic growth rates of the type given in Eqs. (2.5), except here the choice of a is given by $\lambda(2)$. Furthermore, a solution to a linear equation with real eigenvalue cannot make infinite rotations as it approaches the equilibrium, but it must come in along a specific direction. Consequently, the solution for Eqs. (2.9) must exhibit similar behavior, so they do not admit the infinite oscillations as questioned in our version of the Painleve–Wintner question.

For the value $m = \frac{4}{55}$, we can use an argument similar to that given above except that here we use a generalized eigenspace rather than an eigenspace. By the linearization theorems, it is sufficient to analyze the linearized equations. For the solutions possessing nonzero angular momentum, it is easily seen that the rate of convergence is $t^{1/2} \log t$ and that the angle of rotation is finite. Again, a more refined estimate of the type given in Eq. (2.8) can be obtained. See Fig. 2 for a representation of the relationship between η, η', ξ, ξ'; and η, ξ.

Finally, assume that $m > \frac{4}{55}$. A trajectory is either contained in a plane which passes through the x-axis or it includes modes in two such transverse planes which are out of phase. The latter case clearly corresponds to the case for nonzero angular momentum as the solution is noncoplaner. The infinite number of oscillations resulting from the fact that the eigenvalues are complex valued imply that the trajectory has an infinite rotation about the x-axis as s, or as t, goes to infinity. For the scaled equations, the solution approaches the equilibrium point like $\exp(-s/6)$, which means that for the unscaled equations the distance from the particles to the x-axis grows like $t^{1/2}$.

3. The General $\mathbb{N}$ Body Problem

For the n body problem, we will use the coordinates and eigenvalues calculated in [13]. Assume that the center of mass of the system is fixed at the origin of an inertial coordinate system. Let $\mathbf{q}_i$ and $m(i)$ denote the position vector and the mass of the ith particle. With the change of variables $t = \exp(s)$, $\mathbf{Q}_i = \mathbf{q}_i t^{-2/3}$, and ($'$) denoting differentiation with respect to s, the scaled equations of motion are

$$\begin{aligned} \mathbf{Q}_i' &= \mathbf{V}_i, \\ \mathbf{V}_i' &= \tfrac{2}{9}\mathbf{Q}_i + (1/m(i))\,\partial U/\partial \mathbf{Q}_i - \tfrac{1}{3}\mathbf{V}_i, \end{aligned} \tag{3.1}$$

where

$$U = \sum_{i<j} m(i)\,m(j)/|\mathbf{Q}_i - \mathbf{Q}_j|$$

is the self-potential (negative of the potential function) of the system. The central configuration corresponds to equilibria of these equations, i.e., they are positions where

$$\tfrac{2}{9}\mathbf{Q}_i + (m^{-1}(i))\,\partial U/\partial \mathbf{Q}_i = 0, \qquad \mathbf{V}_i = 0 \qquad \text{for all } i. \tag{3.2}$$

We will consider parabolic motion which limits on a collinear central configuration, so we will be interested only in solutions to Eq. (3.2) where the $\mathbf{Q}_i$'s all lie on some straight line. However, notice that since U is invariant with respect to any $SO(3)$ action, so is this equation. This implies that any rotation of a collinear central configuration is again a collinear central configuration; consequently, these configurations are contained in a 2-sphere of fixed points (of Eq. 3.1) in configuration, or phase space. This leads to the original version of the Painleve–Wintner question applied to parabolic orbits [11]; namely, can a parabolic trajectory have more than one of these points as limit points as s goes to infinity? In [13] it was shown for any limiting configuration that the answer is no; each of these trajectories has a single one of these points as a limit point. We start this section by providing a different proof of this fact (Theorem 3), one which uses the additional information available for collinear central configuration; namely, in this case we have some knowledge about the eigenvalues. Furthermore, in the proof of this theorem we will develop the information concerning the eigenvalues which we will need for the remaining results.

A second concern of this section is to see how the mass ratios can affect the rates of convergence of the scaled equations and the growth rates for the unscaled equations. It turns out that the conclusions are somewhat similar to the isosceles case. The statements are given in Theorem 4.

In the "spatial isosceles three body problem with fixed axis" we saw (Theorem 2) that an infinite spin is possible for the more refined version of the Painleve–Wintner problem offered here. Namely, if the angular momentum of the system

$$\mathbf{c} = \sum m(i)\, \mathbf{q}_i \times \dot{\mathbf{q}}_i$$

is nonzero and if the masses satisfy a certain inequality, then the orbit admitted an infinite angle of rotation about the limiting axis of the configuration as t approaches infinity. We will show in Theorem 5 that the same conclusion occurs for the general n-body problem should the angular momentum of the system have a nonzero component in the direction of the limiting axis of the system and if the mass ratios satisfy certain inequalities. This theorem, then, answers the question raised in the Introduction.

Finally, we end this section with some comments concerning conditions on the mass ratios which will admit the oscillatory motion about the limiting axis.

THEOREM 3. *For parabolic orbits tending to a collinear central configuration,* $\mathbf{V}_i$ *tends to zero and* $\mathbf{Q}_i$ *tends to a definite point on the* 2-*sphere of central configurations as s tends to infinity. Furthermore, the behavior of trajectories near any two points on this sphere differ only by the appropriate rotation. For each point on this sphere, the set of solutions of Eq.* (3.1) *which tend to this point as s approaches infinity forms a smooth submanifold of dimension* $5n-7$.

Proof. The linearized equations evaluated at *a* collinear central configuration have the matrix

$$L = \begin{pmatrix} 0 & I \\ \frac{2}{9}I + H & -\frac{1}{3}I \end{pmatrix},$$

where

$$H = \partial/\partial \mathbf{Q}_j[(1/m(i))\, \partial U/\partial Q_i]$$

is the Hessian with weights $m(i)$. All of the eigenvalues of H are real-valued (see [13]). If μ is an eigenvalue of H, then the corresponding eigenvalues of L satisfy the equation

$$\lambda^2 + \tfrac{1}{3}\lambda - \tfrac{2}{9} - \mu = 0$$

or

$$\lambda(+) = -\tfrac{1}{6} + \tfrac{1}{6}\{1 + 36\mu\}^{1/2},$$
$$\lambda(-) = -\tfrac{1}{6} - \tfrac{1}{6}\{1 + 36\mu\}^{1/2}.$$

The $SO(3)$ invariance argument given above to demonstrate that any collinear central configuration gives rise to a 2-sphere of central configurations also implies that H has zero as an eigenvalue with multiplicity two, $\mu(-1) = \mu(0) = 0$. The corresponding eigenvalues for L are $\lambda(-1, +) = \lambda(0, +) = 0$ and $\lambda(-1, -) = \lambda(0, -) = -\frac{1}{3}$. The remaining $3(n-1) - 2 = 3n - 5$ eigenvalues of H are nonzero, [13, pp. 38–42]. The $n - 1$ eigenvalues, $\mu(i)$, $i = -n, \ldots, -2$, corresponding to perturbations along the axis of the collinear central configuration are the same as the eigenvalues for the collinear n-body problem, and they are positive for all choices of the masses. The $2(n-2)$ eigenvalues associated with perturbations off the axis, but orthogonal to the $SO(3)$ action, are negative. The corresponding eigenvalues for L must then satisfy the relationships

$$\begin{aligned}
&\lambda(i, +) > 0, \qquad \lambda(i, -) < -\tfrac{1}{3}, && i = -n, \ldots, -2,\\
&\lambda(i, +) = 0, \qquad \lambda(i, -) = -\tfrac{1}{3}, && i = -1, 0,\\
&\quad -\tfrac{1}{3} < \mathrm{Re}(\lambda(i, +)), \mathrm{Re}(\lambda(i, -)) < 0, && i = 1, \ldots, 2n-4.
\end{aligned} \tag{3.3}$$

These relations on the eigenvalues mean that the equations have an invariant sphere of fixed points where the normal directions to this sphere are hyperbolic. The stable manifold theorem for a normally hyperbolic invariant manifold [4, 6] applies and asserts that the points asymptotic to the sphere form a manifold of dimension $5n - 5$. (This dimension count comes from the $5n - 7$ eigenvalues with negative real part and the dimension of the attracting sphere.) Also, the stable manifold theorem yields that trajectories asymptotic to the sphere are in phase and each is asymptotic to a particular point on the sphere. This last statement completes the proof of the first assertion of the theorem. From the stable manifold theorem we have that the set of trajectories asymptotic to a particular point form a submanifold $W^s(p)$

of dimension $5n-7$. Because of the invariance of the equations under $SO(3)$ action and the uniqueness of the manifolds, the manifolds for different points on the sphere differ by the action of $SO(3)$; i.e., $AW^s(p) = W^s(Ap)$ for A belonging to $SO(3)$. This completes the proof of the theorem.

Incidently, the analysis given above applies to the one isosceles problem not discussed in the previous section; the problem resulting from the involution with the diagonal map with entries $(-1, 1, -1)$. This system admits a circle of central configurations; but if the angular momentum is nonzero, the angular momentum vector must be orthogonal to the axis of the limiting configuration. Furthermore, the system is coplanar iff the angular momentum is zero. The two eigenvalues adjoined to the analysis about the collinear central configuration (due to the added dimension) are 0 and $-\frac{1}{3}$. They represent the rotational symmetry and the angular momentum. It is easy to see that the scaled equations approach the limiting orientation of the axis like $\exp(-s/3)$ and that the system can admit oscillations only if the type described in Theorem 1.

The last statements in the proof of Theorem 3 demonstrate that to understand the behavior of any of these orbits, it suffices to examine the behavior of the orbits tending to one particular limiting central configuration. In particular, we can assume that the limiting collinear central configuration lies on the x, or the $\mathbf{c}$-axis. The following theorem uses this assumption when the different growth rates are determined

THEOREM 4. *For n-body problem, consider the manifold of parabolic orbits which tend to a collinear central configuration. The possible rates of convergence to this expanding configuration,* $t^{2/3}\mathbf{r}(i)$, *are the same as given for the isosceles three body problem where the rates depend upon mass ratios.*

(1) *If the motion is confined to lie on a fixed line, then the convergence is more rapid than* $t^{1/3}$.

(2) *If the configuration formed by the particles is collinear but the line is not fixed, then the distance to the limiting axis is* $t^{1/3}$.

(3) *If the motion is not collinear, then the distance to the limiting axis is* t^b, *where* $\frac{1}{3} < b < \frac{2}{3}$ *and this exponent depends upon the mass ratios.*

Proof. Assume that the limiting collinear central configuration lies on the x-axis. Since the eigenvalues for H along this axis are all positive, it follows from the algebraic equations which determine the eigenvalues for the system that all of the positive and the negative eigenvalues with the most negative values correspond to perturbations along this axis. In particular

$$\lambda(i,+) > 0, \qquad \lambda(i,-) < -\tfrac{1}{3}$$

for $i=-n, \ldots, -2$. This is the same relationship which resulted for the isosceles problem. The eigenvalues $\lambda(i,+)=0$, $i=-1,0$, correspond to tangent vectors to the invariant sphere and $\lambda(i,-)=-\frac{1}{3}$ correspond to a rotation of the axis. It follows from the "in phase" statement in Theorem 3 and the above that not only is there a limiting axis, but that the angle between the configuration and this limiting axis goes to zero like $t^{-1/3}$ for the scaled equations. Of course, for the unscaled equations, this means that the distance from the particles to this limiting configuration due to this particular eigenvalue goes to infinity like $t^{1/3}$.

The remaining eigenvalues for H correspond to directions off of this x-axis, and they have even multiplicity because of the symmetry of the equations about this axis. These eigenvalues, $\mu(i)$, $i=1, \ldots, 2n-4$, result in eigenvalues for L all of which have negative real parts and which satisfy the following relationship:

$$-\tfrac{1}{3} < \operatorname{Re}(\lambda(i,+)), \operatorname{Re}(\lambda(i,-)) < 0.$$

As in the isosceles problem, these eigenvalues can be either real or complex valued depending upon the value of the masses. In particular, if $\mu(i) < -\frac{1}{36}$, then the corresponding pair of eigenvalues $\lambda(i,+)$, $\lambda(i,-)$ have nonzero imaginary parts and their real part equals $-\frac{1}{6}$. On the other hand, should the masses be such that $-\frac{1}{36} \leqslant \mu(i) < 0$, then the corresponding eigenvalues for L are real valued and they satisfy the inequality

$$-\tfrac{1}{3} < \lambda(i,-) \leqslant -\tfrac{1}{6} \leqslant \lambda(i,+) < 0.$$

As a result, if any of the coordinates (η_i, p_i) is nonzero, then $|(\eta_i, \eta_i', \rho_i, \rho_i')|$ tends to zero like $\exp(as)$ or t^a for a equal to the real part of one of the eigenvalues of L. The correspondong result for the unscaled variables is that $|(y_i, y_i', z_i, z_i')| \sim t^{2/3+a}$ with $\frac{1}{3} \leqslant \frac{2}{3}+a < \frac{2}{3}$. (Notice that the equality at $\frac{1}{3}$ corresponds to the eigenvalue $\lambda(0,-)=-\frac{1}{3}$.) Thus, for any mass ratio, the asymptotic growth rates of the distance from the particles to the limiting configuration is t^b where $\frac{1}{3} \leqslant b < \frac{2}{3}$. Consequently, the possible asymptotic rates of approach to the collinear central configuration remain essentially the same as given for the isosceles case, with the slight exception due to the addition of possible rotation of the axis. This completes the proof of this theorem.

THEOREM 5. *Consider parabolic motion which tends to a limiting collinear central configuration, and assume that* $\mathbf{c}$ *is nonzero. If* $\mathbf{u}$ *is a unit vector and* $\mathbf{u}$ *does not lie along the limiting axis, then the rotation about* $\mathbf{u}$ *must be finite. However, if* $\mathbf{u}$ *does lie along the limiting axis and* $\mathbf{c} \cdot \mathbf{u} \neq 0$, *then the system may or may not admit an infinite angle of rotations about this limiting axis; which situation applies is determined by the mass ratios, whether the solution is noncoplanar, and whether the solution lies in a* *manifold determined by a complex-valued eigenvalue. If the system does admit an infinite spin, then the distance from the particles to the limiting axis is* $O(t^{1/2})$.

For the general n-body problem, the distinction between zero and nonzero angular momentum is not given by the planar versus the noncoplanar solutions as there do exist planar solutions which possess nonzero angular momentum. Therefore, we adopt an argument which differs from that used for the isosceles triangle problem to establish the validity of the theorem. Essentially, the idea is as follows: We already have established that the mass ratios influence the growth rate of the system. What we do here is to determine this growth rate orthogonal to the limiting axis. This rate can differ depending on the mass ratios and initial conditions. Next, we relate the growth rate of the system to the angular velocity about the axis determined by u; from these two arguments the conclusion will follow.

Proof. Assume that the angular momentum of the system, $\mathbf{c}$, is nonzero and that the axis determined by $\mathbf{u}$ lies along the x-axis. Let

$$I = \sum m(i)(y^2(i) + z^2(i))$$

denote the moment of inertia about the x-axis, and let $P(\Theta(t))$ from $SO(3)$ be the rotation about the x-axis such that the rotation coordinate system $P(\Theta(t))^{-1}\,(x, y, z)$ has no component of the angular momentum along the x-axis. If we denote the derivative of Θ with respect to t as w, then, as we will show at the end of this proof, it follows that

$$c(1) = wI, \tag{3.4}$$

where $c(1)$ is the constant component of $\mathbf{c}$ in the $\mathbf{u}$ direction.

LEMMA. *If* $c(1) \neq 0$, *then the system admits an infinite spin about the axis defined by* $\mathbf{u}$ *if and only if*

$$\int^{\infty} I^{-1}\,dt = \infty.$$

In particular, the system has an infinite spin if $I = O(t \ln t)$, *but it has a finite spin if* I *grows fast enough so that*

$$I^{-1} = O(t^{-1} \ln^{-2} t).$$

By solving Equation 3.4 for w, and then integrating the resulting equality, we find that

$$\lim_{T \to \infty} (\Theta(T) - \Theta(1)) = \lim_{T} \int_1^T w\,dt = \lim_{T} \int_1^T c(1)\,I^{-1}\,dt.$$

The assertion of the Lemma follows from this relationship. Thus, the conclusion concerning whether or not the system admits an infinite spin depends upon the growth rate of I.

From the lemma, we can establish that for infinite spin to occur about the axis defined by $\mathbf{u}$, this axis and the limiting axis must coincide. This is because if they do not, then for some choice of i, $\mathbf{Q}_i$ approaches a limiting position off of the x-axis. Thus the unscaled growth rate of the system orthogonal to the x-axis becomes $t^{2/3}$, or I grows like $t^{4/3}$. The conclusion now follows from what we have asserted about the growth rate of I and w.

Assume that the limiting position of the axis is the x-axis. If we are to have an infinite spin, then it follows from what we have established in Theorem 4 that the solutions will have to lie in certain submanifolds. If the solution lies in any of the submanifolds corresponding to eigenvectors off of the x-axis direction and determined by eigenvalues greater than $-\frac{1}{6}$, then it follows from Hartman's C^1 linearization result that I grows sufficiently fast to satisfy the lower bound growth rate given in the lemma, so only a finite rotation obtains. On the other hand, suppose that corresponding to these other direction, the solution lies in a submanifold determined by eigenvalues all strictly smaller than $-\frac{1}{6}$. Because the eigenvalues corresponding to the x-axis are all real valued, it follows that such a solution must lie in a manifold determined by real-valued eigenvalues. The corresponding linearized solution for the scaled equations must approach the equilibrium point with a definite angle. Consequently, according to the Hartman result, the solution must approach the equilibrium point without infinite rotation. This, of course, implies that the unscaled equations do not admit an infinite spin. However, these assumptions imply that $I = O(t)$, so for this to be compatible with Eq. (3.4), the solutions must admit zero value for $c(1)$. (Actually, it turns out that for such solutions, $\mathbf{c} = \mathbf{0}$.)

Now assume that the mass ratios are such that the maximum eigenvalue is $-\frac{1}{6}$, where, from what was shown above, we have that this eigenvalue must have multiplicity at least 2. Since the eigenvalues are all real valued the same finite rotation conclusion applies. Anyway, if the solution yields behavior like $t \ln^2(t)$, the growth rate of I satisfies the lower growth rate inequality given in the lemma, and finite rotation obtains.

Finally, assume that the mass ratios are such that some eigenvalue has a nonzero imaginary part. The real part is equal to $-\frac{1}{6}$. In this case, by restricting attention to solutions belonging to the submanifold corresponding to this eigenvalue and others with real part not larger than $\frac{1}{6}$, it follows from the linearization theorems and the above statement concerning the growth of I that the solution admits infinite spin and that the growth rate of the particles from the limiting axis is like $t^{1/2}$. The existence of such solutions, indeed a manifold of such solutions, is obvious should the appropriate mass ratios exist.

What remains is to establish Eq. (3.4). The motivation for what follows comes from the process of elimination of the node, finding the effective potential, and the reduced phase space. (See, [1, Sects. 4.3 and 4.5].) Although we do not use any of these theorems, the approach indicates what to search for. Also, compare the following with [13, pp. 36–38]. What we need is an angle $\Theta(t)$ such that a rotation about the x-axis through the angle $\Theta(t)$, $P(\Theta(t))$, eliminates the angular momentum. Let vector $\mathbf{w} = \Theta'\mathbf{u}$ and vector $\mathbf{f}_i(t) = P(\Theta(t))^{-1}\mathbf{q}_i(t)$. (Here, $\mathbf{u}$ is the unit vector in the positive x direction.) Then

$$\mathbf{q}_i' = P\mathbf{f}_i' + \mathbf{w} \times \mathbf{q}_i$$

and

$$\begin{aligned} c(1) &= \mathbf{u} \cdot \mathbf{c} = \mathbf{u} \cdot \sum m(i)\, \mathbf{q}_i \times \mathbf{q}_i' \\ &= \mathbf{u} \cdot P \sum m(i)\, \mathbf{f}_i \times \mathbf{f}_i' + \sum m(i) \{\mathbf{q}_i \times (\mathbf{w} \times \mathbf{q}_i)\} \cdot \mathbf{u}. \end{aligned}$$

If $\mathbf{w}$ can be selected so that the second term equals $c(1)$, then

$$0 = \mathbf{u} \cdot P \sum m(i)\, \mathbf{f}_i \times \mathbf{f}_i' = \mathbf{u} \cdot \sum m(i)\, \mathbf{f}_i \times \mathbf{f}_i'.$$

The last equality follows because matrix P preserves the $\mathbf{u}$ direction. Now, if we let $\mathbf{r}_i = (0, \mathbf{y}_i, \mathbf{z}_i)$, we have

$$\begin{aligned} \mathbf{u} \cdot \sum m(i)\, \mathbf{q}_i \times (\mathbf{w} \times \mathbf{q}_i) &= \mathbf{u} \cdot \sum m(i)\, \mathbf{q}_i \times (\mathbf{w} \times \mathbf{r}_{i_2}) \\ &= \mathbf{u} \cdot \sum m(i)\, \mathbf{r}_i \times (\mathbf{w} \times \mathbf{r}_i) \\ &= \mathbf{u} \cdot \mathbf{w} \sum m(i)\, |r_i|^2 = wI. \end{aligned}$$

So, to find $\Theta(t)$, we let $w = I^{-1}c(1)$ and we define $\Theta(t) = \int_0^t w\, dt$. This completes the proof.

Finally, it can be shown that for any choie of $n > 2$, there are mass ratios such that the complex eigenvalues required to lead to this type of motion do exist. We shall briefly outline how this can be done for $n = 3$. The conditions for such an eigenvalue to occur are that some eigenvalue of matrix H is less than $-\frac{1}{36}$. The eigenvalues of H are analytic functions of the masses and the equilibrium positions, and the equilibrium conditions are in turn smooth functions of the masses. Thus, the eigenvalue condition can be set to $-\frac{1}{36}$ to find the surface separating those mass ratios which do admit oscillatory motion from those which do not. For the three body problem, we normalize

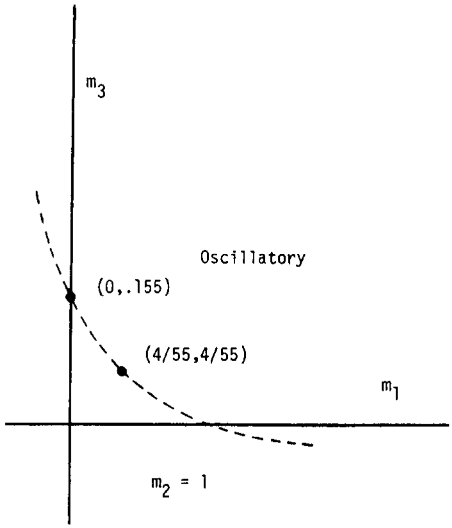

FIG. 3. Bifurcation curve between oscillatory and nonoscillatory motion.

the masses by assuming that the middle one $m(2) = 1$. Figure 3, which was obtained by computer computations, gives the curve for the mass ratio, where the larger region corresponds to mass ratios which admit oscillatory behavior. What follows is a brief discussion establishing that the essential features of this bifurcation curve are indeed correct.

Following Pollard [9], for a collinear central configuration, let the distance between particles 1 and 2 be given by a and the distance between 2 and 3 be given by ar. In this way r becomes a scale factor comparing the two relative distances, and its value is given by the unique positive zero of the equation

$$F(m(1), m(3), r) = (1 + m(3)) + (2 + 3m(3))r + (1 + 3m(3))\, r^2 - (1 + 3m(1))\, r^3 - (2 + 3m(11)\, r^4 - (1 + m(1))\, r^5 = 0. \tag{3.5}$$

When evaluated at a collinear central configuration, the eigenvalues of the system admit nonzero imaginary parts iff the masses are such that the following polynomial has a positive value, where the value of r comes from Eq. (3.5):

$$G(m(1), m(3), r) = (8m(3) - 1) + (24m(3) - 2)r + (23m(1) + 23m(3) - 2)\, r^2 + (24m(1) - 2)\, r^3 + (8m(1) - 1)\, r^4 \tag{3.6}$$

Notice that whenever both masses are greater than or equal to $\frac{1}{8}$, then we obtain eigenvalues with imaginary parts. This is because with these bounds, all of the coefficients of G are nonnegative, so G is positive valued for any $r > 0$.

The curve in Fig. 3 is obtained by solving for the zero set of Eqs. (3.6) and (3.5). The properties of this zero set are briefly characterized in

THEOREM 6. *Let C be the curve in the positive quadrant of $m(1)$–$m(3)$ space determined by the zero set of Eqs.* (3.5), (3.6). *Set C is a smooth algebraic curve parameterized by r with the property that $m(3)$ increases and $m(1)$ decreases with increasing values of r, and any ray in the positive quadrant corresponding to a fixed ratio of the two masses intersects set C in precisely one point. Finally, if both masses are strictly bounded below by $\frac{4}{55}$, then G is positive, and the three-body system admits complex eigenvalues.*

The proof of this theorem is highly computational, so we only indicate the steps which must be taken.

Proof. That set C is algebraic follows immediately from the fact that the two defining equations are polynomials. That this algebraic set is smooth and parameterized by r follows directly from the implicit function theorem. Here the four entries in the Jacobean matrix obtained by taking the partial derivatives with respect to the two masses yield four polynomials in r with constant coefficients. For three of the polynomials, the coefficients are all nonnegative; for the fourth they are all nonpositive. The nonsingularity of the determinant follows immediately for $r > 0$. That this set consists of a single curve and that this curve has monotone properties in that it cannot fold back on itself will follow once we show that each ray intersects the set in one and only one point. However, first we consider the growth properties of the curve.

According to the De Cartes rule of signs, it follows that F has precisely one positive real root, $r(L)$; furthermore, F is negative for large values of r. Therefore, $F'(r(L)) < 0$ where ($'$) denotes the partial derivative with respect to r. By exploiting the symmetry of the problem, it is only necessary to consider the case where $r(L) > 1$. It is well known and easy to establish that $r(L) > 1$ iff $m(3) > m(1)$ and that $r(L) = 1$ iff $m(3) = m(1)$. We will assume that these inequalities apply for what follows. A direct computation shows that if $r(L) = 1$, then both masses on set C must equal $\frac{4}{55}$.

Again, by use of De Cartes rule of sign, G admits at most two positive roots depending upon whether the masses are bounded above by $\frac{1}{8}$. Furthermore, $G(\frac{4}{55}, \frac{4}{55}, r)$ has a double zero at $r = 1$, and if $m(1) < \frac{1}{8}$, G is negative valued for large values of r. Consequently, as we vary the masses, the graph of G will either pull away from the axis (corresponding to G being always negative), the graph will cross the axis giving two zeros, or the graph will be shifted so that a double zero still occurs. In any case, on set C, $G'(r(L)) < 0$, where equality obtains iff $R(L)$ corresponds to a multiple zero of G.

To determine how $\dot{m}(1)$ and $\dot{m}(3)$ change on C with r, where $(\dot{\ })$ denotes the derivative with respect to r, differentiate both Eqs. (3.5) and (3.6) with respect to r (where we set $G=0$) and solve. By doing this we obtain

$$\begin{pmatrix} \dot{m}(1) \\ \dot{m}(3) \end{pmatrix} = P \begin{pmatrix} 8+24r+23r^2 & -1-3r-3r^2 \\ -23r^2-24r^3-8r^4 & -3r^3-3r^4-r^5 \end{pmatrix} \begin{pmatrix} F' \\ G' \end{pmatrix},$$

where P is a positive-valued polynomial. From this and what we have established about the signs of F' and G' at $r(L)$, it follows immediately that in some neighborhood of $r(L)=1$, $m(1)$ is decreasing and $m(3)$ is increasing with respect to r. Once we show that the magnitude of G' is sufficiently small for values of the masses from C in the positive quadrant, then this statement extends to all of C (we will indicate how to show this below), but in any case we have that $m(3)$ is an increasing function.

Next, we consider a ray in $m(1)$–$m(3)$ space. Let $m(1)=m$ and $m(3)=tm$, where $t>1$ is a fixed constant but m can vary. The zero set $F(m, tm, x)=0$ determines how the equilibrium position $r(L)$ varies with m. It is easy to show that the graph of this set on an (m, x) coordinate system passes through the point $(0, 1)$ and monotonically approaches the value $x=r(T)$ as m approaches infinity where $r(T)$ is the unique zero of the polynomial $t\{1+3r+3r^2\}-\{3r^3+3r^4+r^5\}$. On the other hand, the zero set of $G(m, tm, x)$ lies in the interval $0<m<\frac{1}{8}$, it passes through the point $(\frac{1}{8}t, 0)$, it starts off to the left, it has at most two values of x for each value of m, and it approaches infinity as m approaches $\frac{1}{8}$. It is clear that these two curves must intersect and that any intersection must occur inside the box $0<m<\frac{1}{8}$, $1<x<r(T)$; such an intersection corresponds to a point on set C. To show that these curves cannot intersect more than once, we compare the gradients of G and F at any such intersection. As we have shown above, the partial of both functions with respect to x is negative, and a direct computation shows that the partial with respect to m is positive. Therefore, both gradients are in the same quadrant. However, a direct computation using the given bounds on m and x, demonstrates that there is always a positive acute angle from the gradient of F to that if G. This completes the proof.

Incidently, these estimates use the upper bound on m to show that G' is "small." The same estimates complete the proof that $m(1)$ decreases with increasing $m(3)$ on C.

4. McGehee Coordinates

In this section we wish to indicate how the above results could have been obtained by use of the McGehee coordinates as developed in [2] and [3].

Let $\mathbf{q} = (\mathbf{q}_1, ..., \mathbf{q}_n)$ be the location of the particle in configuration space, where we assume that the center of mass of the system is fixed at the origin. Let $\mathbf{p} = (m(1)\,\mathbf{q}_1', ..., m(n)\,\mathbf{q}_n')$ be the momentum vector. Let $M = \text{diagonal}(\mu(i))$ be the $3n \times 3n$ matrix of masses, where $m(i) = \mu(3i - 3 + j)$ for $j = 1, 2, 3$. Let V be the potential energy (the negative of the self potential which was used in Section 3). The equations of motion then become

$$\mathbf{q}' = M^{-1}\mathbf{q}, \qquad \mathbf{p}' = -\partial V/\partial \mathbf{q}.$$

The McGehee coordinates given in [3] (also see [2, 10]) are found by dividing by the square root of the moment of inertia, $r = (\mathbf{q}^t M \mathbf{q})^{1/2}$. The scaled configuration is $\mathbf{s} = r^{-1}\mathbf{q}$, where $\mathbf{s}$ lies on the ellipsoid $\mathbf{s}^t M \mathbf{s} = 1$. The momentum is decomposed into a radial component along $\mathbf{s}$ and the component tangent to the ellipsoid. Then each component is scaled by $r^{1/2}$ to obtain

$$v = r^{1/2}\mathbf{s}^2\mathbf{p} \qquad \text{and} \qquad \mathbf{u} = r^{1/2}(M^{-1}\mathbf{p} - y\mathbf{s}).$$

It turns out that the differential equation for r, v, $\mathbf{s}$, $\mathbf{u}$ all have a factor of $r^{3/2}$, so the change of the time scale $d\tau/dt = r^{-3/2}$ yields the equations

$$\begin{aligned} r' &= rv, \\ v' &= \mathbf{u}^t M \mathbf{u} + \tfrac{1}{2}v^2 + V(\mathbf{s}) \\ \mathbf{s}' &= \mathbf{u} \\ \mathbf{u}' &= -\tfrac{1}{2}v\mathbf{u} - (\mathbf{u}^t M \mathbf{u})\mathbf{s} - V(\mathbf{s})\mathbf{s} - M^{-1}\nabla V(\mathbf{s}). \end{aligned} \tag{4.1}$$

(For details, see [3].)

To study parabolic orbits, r goes to infinity, se set $\rho = (1/r)$. The only equation which changes is the first which now becomes $\rho' = -\rho v$. These new equations extend naturally to $\rho = 0$ to give the motion at infinity. The energy relation is

$$(1/\rho)e = \tfrac{1}{2}\mathbf{u}^t M \mathbf{u} + \tfrac{1}{2}v^2 + V(\mathbf{s}),$$

where $e = 0$ for parabolic orbits.

The parabolic orbits which tend to $\rho = 0$ can be shown to approach the fixed points of these equations, and they correspond to the central configurations: $\rho = 0$, $v = 0$, and $\mathbf{s}(0)$, where $\mathbf{s}(0)$ is such that it is a scalar multiple of $M^{-1}\nabla V(\mathbf{s}(0))$. The term $v(0)$ is obtained from the energy relation. Proceeding as in Section 3, one can establish that there is a simple relationship between the eigenvalues of B, the linear part of $M^{-1}\nabla V(\mathbf{s}) + V(\mathbf{s})\mathbf{s}$ evaluated at $\mathbf{s}(0)$, and the eigenvalues of Eq. (4.1). It turns out that if μ is an eigenvalue of B and $\lambda(-)$ and $\lambda(+)$ are the corresponding eigenvalues of the linearized equations (5.1), then

$$\lambda(+), \lambda(-) = -\tfrac{1}{4}v(0) \pm \tfrac{1}{4}\{1 - 16\mu v(O)^{-2}\}^{1/2}$$

(See [3].) The perturbations off the axis of the collinear configuration have $\mu_i > 0$, so

$$-\tfrac{1}{2}v(0) < \lambda(i, -) \leqslant -\tfrac{1}{4}v(0) \leqslant \lambda(i, +) < 0.$$

These limits differ from those given in Section 3 because of the difference in the scaling of the independent variable time; however, the final conclusions remain the same as we show below.

From the linearized equation of (4.1), it follows that a parabolic orbit will satisfy $\rho \sim \exp(-v(0)\tau)$. Since $dt/d\tau = \rho^{-3/2} \sim \exp(\tau b)$, where $b = 3v(0)/2$, it follows that $t \sim \exp(b\tau) \sim \rho^{-3/2}$. In other words, $\rho \sim t^{-2/3}$. This growth rate, which was used in the beginning of Section 3, comes out after the fact here.

The growth of the other coordinates resulting from the eigenvalues $\lambda(i, \pm) = av(0)$ satisfy

$$r \exp(\tau a v(0)) \sim (t^{2/3})^{1+a} = t^b$$

with $b = \tfrac{2}{3}(1 + \lambda(i, \pm)/v(0))$. Consequently,

$$-\tfrac{1}{2}v(0) < \lambda(i, -) \leqslant -\tfrac{1}{4}v(0), \qquad \tfrac{1}{3} < b < \tfrac{1}{2},$$
$$-\tfrac{1}{4}v(0) \leqslant \lambda(i, +) < 0, \qquad \tfrac{1}{2} \leqslant b < \tfrac{2}{3}.$$

Thus, as one would hope, the growth rates remain the same as given in Section 3 by use of the other coordinate system.

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